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The Timing of Bets and the Favorite-Longshot Bias*

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Abstract

In parimutuel betting markets, it has been observed that proportionally too many bets are placed on longshots, late bets are more informative than early bets, and a sizeable fraction of bets are placed early. We propose an explanation for these facts based on equilibrium incentives of privately informed rational bettors, who profit from betting against bettors with recreational motives. We show that small rational bettors who act on private information have an incentive to wait until the last minute, and then bet without access to the information of the others. Once the distribution of bets is revealed, the longshot is recognized to be less likely to win than was originally thought. When acting on common information instead, bettors have an incentive to place early bets in order to preempt others from exploiting the same information.

Keywords: Parimutuel betting, favorite-longshot bias, private information, timing.

JEL Classification: D82 (Asymmetric and Private Information), D83 (Search; Learning; Information and Knowledge), D84 (Expectations; Speculations), G13 (Contingent Pricing; Futures Pricing).

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1. Introduction

According to the market efficiency hypothesis, the price of a financial asset is an unbiased estimate of its fundamental value, reflecting all the information available to the market. It has proven difficult to perform direct tests of this hypothesis, due to the lack of exogenous measures of fundamental values of the assets traded in regular financial markets.¹ Researchers have turned to betting markets, where fundamental values are both observable and typically exogenous, and have found a number of puzzling empirical facts. In this paper, we propose a theoretical explanation for these regularities.

Our analysis is based on the institutional features of *parimutuel betting* markets.² These are mutual markets, in which the total money bet on all outcomes (net of the *track take*)³ is shared proportionally among those who bet on the winning outcome. Typically, bets are placed in real time, resulting in provisional odds that are publicly displayed and updated at regular intervals until *post time*, when betting is closed. Since the payments are made exclusively on the basis of the final distribution of bets, individual bettors do not know with certainty the odds they face.⁴

In the context of a horse race, market efficiency predicates that the final distribution of parimutuel bets is directly proportional to the market’s assessment of the horses’ chances of winning. This is because the gross expected payoff of a bet on a horse is equal to the ratio of its probability of winning to the proportion of bets placed on that horse. The expected payoffs on the different horses are equalized when the fraction of money bet on each of them is equal to the probability that the horse wins.

Starting with Griffith (1949), horse race betting data have been used to test this proposition. The proportion of money bet on a horse has been shown to track closely its em-

¹In regular financial markets, it is typically impossible to observe the fundamental values of traded assets. In addition, the performance variables used as proxies of value are not exogenous, being themselves affected by market prices.

²Since its introduction in the nineteenth century, parimutuel betting has become the most common wagering system at racing tracks throughout the world. Parimutuel betting is commonly adopted at horse and greyhound racing tracks, as well as for soccer, basketball, jai alai, and other games. As further discussed in Section 6, the parimutuel structure has been recently used in financial markets to allow traders to hedge risk related to economic statistics, such as employment, retail sales, and industrial production.

³The track take includes taxes and other charges for the expenses of running the race and the betting scheme.

⁴It is worth contrasting parimutuel betting with the alternative scheme of “fixed odds” betting (cf. Dowie 1976). In fixed odds betting, bookmakers accept bets at specific, but changing, odds throughout the betting period. In this case, the return to any individual bet is therefore not affected by the bets placed subsequently. In the UK, parimutuel and fixed odds betting coexist.

pirical chance of winning, in support of the market efficiency hypothesis. However, three somewhat puzzling regularities have emerged:

1. Horses with short odds (i.e., favorites) tend to win even more frequently than indicated by the final market odds, while horses with long odds (i.e., longshots) win less frequently (see e.g., Thaler and Ziemba 1988). This is known as the *favorite-longshot bias*.
2. Late bets tend to contain more information about the horses' finishing order than earlier bets (Asch, Malkiel and Quandt 1982). This is the phenomenon of *late informed betting*.
3. A large amount of money is placed just before post time, but sizeable amounts are placed much earlier (Camerer 1998).⁵ If more information becomes available later, why are so many bets placed well before post time? This is the puzzle of *early betting*.

In this paper, we formulate a simple theoretical model that sheds light on these facts. The model posits an exogenous initial distribution of bets placed by *outsiders*, so that the rational *insiders* can earn non-negative returns despite the presence of a positive track take. Each insider has some private information, modeled as an informative signal about the outcome of the race. These informational assumptions are similar to those made in the market microstructure literature, and applied to fixed odds betting by Shin (1991 and 1992). We focus instead on parimutuel betting, and manage to obtain a tractable model by assuming that private information is a continuous variable, as is commonly done in auction theory. Koessler and Ziegelmeyer (2002) is the only other paper in the literature that analyzes parimutuel betting under asymmetric information.⁶

Our explanation of the favorite-longshot bias is based on the informational content of the distribution of bets. We argue that the presence of private information introduces a systematic wedge between the final distribution of bets and the market's beliefs. To

⁵In Camerer's data set roughly half of the money is placed in the three minutes before post time, and half (often well) before then.

⁶In Koessler and Ziegelmeyer's model, bettors have binary signals and bet sequentially with exogenous order. We instead analyze simultaneous betting, derive the favorite-longshot bias, and offer insights about the forces driving the endogenous timing of bets. See also Feeney and King (2001) for the characterization of the equilibria in a sequential parimutuel game with complete information and exogenous order.

understand this, suppose for the moment that some privately informed insiders bet simultaneously at post time. When many (few) insiders end up betting on the same outcome, which now becomes more of a favorite (longshot), it means that many (few) had private information in favor of this outcome. If individual bettors knew that many bets would have been placed on that outcome, they would have been even more likely to bet on that outcome. But the final distribution of bets is not known when betting, so that this inference can only be made after betting is closed. Intuitively, a disproportionately low (high) fraction of bets are placed on favorites (longshots), because these bets were placed without knowing the final odds (Proposition 2). Despite its simplicity, this explanation of the favorite-longshot bias has not been proposed before.⁷

We argue that the timing of bets is driven by two opposite forces. On the one hand, there is an incentive to wait in order to hide one's *private information* and possibly see that of others. On the other hand, there is an incentive to bet early in order to prevent others from exploiting the *market power* stemming from the public information shared with other strategic bettors.

Our explanation of the second regularity is based on private information. We show that when the inside are “small”, in the sense that they are price takers, this first effect dominates and bets are simultaneously placed at post time. This outcome (Proposition 5) formalizes a prevailing intuition, initially formulated by Asch, Malkiel and Quandt (1982).

Our explanation of the third regularity is based on market power. To understand this, consider a small number of bettors who share the same information about the horses' winning chances, and so are not concerned about revealing this information. Due to the parimutuel structure, the payoff per dollar bet on a profitable horse is a decreasing function of the total bets placed on that horse. Bettors are effectively competing in a market with a downward sloping demand curve, with the final price determined by the final distribution of bets. This market power channel introduces an incentive to bet early, in order to prevent competitors from unfavorably changing the odds against them (Proposition 6). This strategic incentive to place early bets is based on the simple fact that parimutuel betting is a Cournot game, and appears not have been noticed before.

The paper proceeds as follows. After formulating the model in Section 2, in Section 3

⁷Our companion paper (Ottaviani and Sørensen 2004) contains an additional analysis of this informational explanation. See Section 3.2 for a discussion of the other explanations proposed in the literature on the favorite-longshot bias.

we focus on the simple case of simultaneous betting in the last period. After showing that the model predicts the favorite-longshot bias, we obtain testable results on how its extent depends on the amount of pre-existing bets, the level of the track take, and ratio of the size of the population of informed to uninformed bettors. We then endogenize the timing, by allowing the bettors to decide when to place their bets. In general, bets not only reveal information to other bettors but also affect odds. We analyze these two effects in isolation by considering two versions of the model in turn. Section 4 considers a continuum of small informed bettors, who in equilibrium postpone their bets to the end. In contrast, Section 5 shows that early betting results when bettors affect the market odds, but are not concerned about revealing information. We conclude in Section 6 by discussing the predictions of our theory and some avenues for future research. The proofs are collected in the Appendix.

2. Model

We present a stylized model of parimutuel betting on the *outcome* of a race between two horses, A and B . The winning horse is identified with the *state*, $x \in \{A, B\}$. Time is discrete and betting is open in a commonly known finite window of time, with periods denoted by $t = 0, 1, \dots, T$.⁸ Betting opens at $t = 1$ and closes at $t = T$ (post time). A publicly observable tote board displays in any period the cumulative amounts bet until then on each horse.

At the race track, some bets are placed for recreational purposes based on idiosyncratic preferences for particular horses, while others are motivated by profit maximization. Even though in reality individual bettors could be motivated by a combination of recreation (private value) and profits (common value), for convenience our model separates recreational from profit-maximizing bettors. In this way, we depart from the literature on preferences for risk taking, and can conveniently allow for the presence of private information.⁹

The amount of exogenously given bets placed on outcome x at time $t = 0$ by unmodeled *outsiders* is denoted by $n(x)$. The outsiders play a role similar to liquidity (or noise) traders

⁸Typically, betting is open for a period before the beginning of the race. The provisional odds are publicly displayed on a board at regular intervals until betting is closed and the race starts. For example, the UK's Tote updates the display every 30 seconds. The assumption of discrete time is technically convenient, but is not essential for our results.

⁹A similar approach is commonly adopted in market microstructure models, where liquidity traders are separated from informed arbitrageurs.

in models of financial markets. For simplicity, we assume that the outsiders' bets are not random.

There is a continuum $[0, I]$ of rational bettors (or *insiders*).¹⁰ These bettors have private information (see Section 2.1), are risk neutral, and maximize their expected return.¹¹ Each bettor faces an identical wealth constraint, being able to bet an amount normalized to 1. Bettors decide if, when, and whether to bet on $x = A$ or $x = B$. Note that the presence of the outsiders allows the insiders to derive positive expected payoff from betting. The total amount bet by insiders on outcome x is denoted by $m(x)$.

We assume that bets cannot be cancelled once they are made.¹² The total amount bet by insiders and outsiders is placed in a pool, from which a proportional *track take* τ is taken. The remaining money is then evenly distributed to the winning bets. If x is the winner, each unit bet on outcome x yields

$$(1 - \tau) \frac{\sum_{y=A}^B (n(y) + m(y))}{n(x) + m(x)}.$$

Since each bettor is small, these payoffs are not affected by any individual's bet. This price taking assumption is the essential feature of our continuum population assumption.¹³ The insiders know the exact amounts $n(A)$ and $n(B)$ and the other parameters of the model.

2.1. Information

Private information is believed to be pervasive in horse betting (see e.g., Crafts 1985). It is modeled as follows. The insiders are assumed to share a common *prior belief* $q = \Pr(A)$ that horse A is the winner, possibly formed after the observation of a common signal. At $t = 0$, before betting begins, each bettor i is assumed to privately observe a *signal* s_i . Conditionally on state x , these signals are assumed to be identically and independently distributed with probability density function (p.d.f.) $f(s|x)$.

¹⁰See Watanabe (1997) for an alternative model of parimutuel betting with a continuum of bettors. In his model, the bettors' heterogeneous beliefs are assumed to be common knowledge. In our model instead, beliefs reflect information about the state.

¹¹Payoffs are not discounted, since betting takes place within a short time frame.

¹²Typically, patrons are not allowed to cancel their bet after leaving the seller's window. More generally, there are serious limitations on the right to cancel a bet, unless it is due to a mistake. These limitations are set in order to prevent manipulation and preserve integrity of the wagering process. See e.g., Canada Department of Justice (1991), paragraph 57, and Delaware Harness Racing Commission (1992), rule 9.4.1.3.

¹³With a continuum of bettors, each of them takes the price as given. See our companion paper Ottaviani and Sørensen (2004) for an analysis of simultaneous betting with a finite number of insiders. We also depart from the price taking assumption in Section 5 when studying the effect of market power.

Upon observation of signal s , the prior belief q is updated according to Bayes' rule into the *posterior belief* $p = \Pr(A|s) = qf(s|A) / (qf(s|A) + (1 - q)f(s|B))$. This transformation of s into p determines the conditional distributions of p given x from the corresponding distributions of s given x . The posterior belief p is then distributed according to the cumulative probability distribution function (c.d.f.) G on $[0, 1]$. By the law of iterated expectations, G must satisfy $q = E[p] = \int_0^1 p dG(p)$.¹⁴

The conditional distributions of p given x play a central role in our analysis. These p.d.f.s are given by $g(p|A) = pg(p)/q$ and $g(p|B) = (1 - p)g(p)/(1 - q)$, since Bayes' rule yields $p = qg(p|A)/g(p)$ and $1 - p = (1 - q)g(p|B)/g(p)$. The monotonicity in p of the likelihood ratio $g(p|A)/g(p|B) = (p/(1 - p))((1 - q)/q)$ reflects the property that higher beliefs in outcome A are relatively more likely when outcome A is true. This monotonicity implies that $G(p|A)$ first-order stochastically dominates $G(p|B)$, so that $G(p|A) < G(p|B)$ for all $p \in (0, 1)$.

Throughout, we assume that G is *continuous* with p.d.f. g ,¹⁵ and has *full support* equal to the beliefs set $[0, 1]$.¹⁶ Some results are derived under the additional assumption that the posterior distribution is *symmetric*, i.e. that $G(p|A) = 1 - G(1 - p|B)$, so that the chance of posterior p conditional on state A is equal to the chance of posterior $1 - p$ conditional on state B .

Our results are conveniently illustrated by the *linear signal example* with conditional p.d.f. $f(s|x = 1) = 2s$ and $f(s|x = -1) = 2(1 - s)$ for $s \in [0, 1]$, and corresponding c.d.f. $F(s|x = 1) = s^2$ and $F(s|x = -1) = 1 - (1 - s)^2$. This signal structure can be derived from a binary signal with uniformly distributed precision. With fair prior $q = 1/2$, we have $p = s$ so that $G(p|A) = p^2$ and $G(p|B) = 2p - p^2$.

3. Betting in the Last Period

We aim to prove that insiders postpone their bets until the last period, and to characterize the amounts of final insider bets on the two horses. In this section, we analyze the

¹⁴While the conditional distributions of the signal is our primitive, there is no loss of generality in making assumptions directly on G . Any posterior belief distribution with $q = E[p]$ can be derived from the assumption that the distribution of the signal s is the same as the distribution of the posterior p .

¹⁵In the presence of discontinuities in the posterior belief distribution, equilibria might involve mixed strategies. Our results can be extended to allow for these discontinuities.

¹⁶This implies that there exist arbitrarily informative signal realizations, so that we conveniently obtain an interior equilibrium.

simultaneous betting game that takes place in the last period. In Section 4, we then solve the full dynamic game by using a backwards induction argument.

For the purpose of this section, bets committed by insiders before period $t = T$ are considered part of the outside bets $n(A), n(B)$. The population size I refers to the continuum of insiders who did not bet before the last period, and G is the c.d.f. of their posterior beliefs. We maintain the assumption that G has full support.¹⁷

In Section 3.1 we show that there exists a unique equilibrium in period T , and fully characterize the equilibrium strategies. In Section 3.2 we show that the equilibrium exhibits the favorite-longshot bias, and in Section 3.3 we derive some testable comparative statics results.

3.1. Equilibrium

In a Bayes-Nash equilibrium, every rational bettor best replies to a correctly predicted fraction of the insiders who bet on each outcome in each state. Denote by $m(y|x)$ the amount bet by the insiders on outcome y when state x is true. If state x is true, the gross payoff to a bet on outcome x is

$$W(x|x) = (1 - \tau) \frac{n(A) + n(B) + m(A|x) + m(B|x)}{n(x) + m(x|x)} > 0. \quad (3.1)$$

Consider the decision problem of a bettor with belief p . The expected return from a unit bet on outcome A is $pW(A|A) - 1$. The expected payoff for a bet on B is $(1 - p)W(B|B) - 1$, and for not betting is 0. It then follows immediately that there exist thresholds $\hat{p}_B, \hat{p}_A \in [0, 1]$ such that the bettor optimally bets on B when $p < \hat{p}_B$, abstains when $\hat{p}_B < p < \hat{p}_A$, and bets on A when $p > \hat{p}_A$.

If the winning probability of horse A implied by the pre-existing bet is not too extreme compared to the track take, in the unique equilibrium insiders bet on both outcomes.

Proposition 1 *Assume that*

$$0 \leq \tau < \min \left\{ \frac{n(A)}{n(A) + n(B)}, \frac{n(B)}{n(A) + n(B)} \right\}. \quad (3.2)$$

There exists a unique Bayes-Nash equilibrium of the last-period game. Every insider bets on B when $p < \hat{p}_B$, abstains when $\hat{p}_B < p < \hat{p}_A$, and bets on A when $p > \hat{p}_A$, where

¹⁷We argue below that the case in which G is no longer unbounded ($0 < G(p) < 1$ for all $p \in (0, 1)$) results in a rather trivial last-period game.

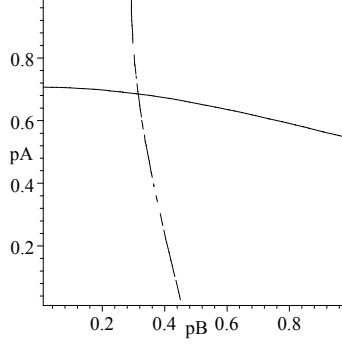


Figure 3.1: This illustrates the determination of equilibrium in the linear signal example introduced in Section 2, with $q = 1/2$, $n(A) = n(B) = I$, and $\tau = .15$. The solid line represents the set of (\hat{p}_B, \hat{p}_A) solving (3.3), while the dashed line represents the solutions to (3.4). Both curves are downward sloping, and the solid line crosses from below the dashed line at the equilibrium values of (\hat{p}_B, \hat{p}_A) .

the thresholds $0 < \hat{p}_B < \hat{p}_A < 1$ constitute the unique solution to the two indifference conditions

$$\hat{p}_A = \frac{1}{1 - \tau n(A) + n(B) + I(1 - G(\hat{p}_A|A)) + IG(\hat{p}_B|A)} \frac{n(A) + I(1 - G(\hat{p}_A|A))}{1} \quad (3.3)$$

and

$$\hat{p}_B = 1 - \frac{1}{1 - \tau n(A) + n(B) + I(1 - G(\hat{p}_A|B)) + IG(\hat{p}_B|B)} \frac{n(B) + IG(\hat{p}_B|B)}{1}. \quad (3.4)$$

Proof. See the Appendix. □

Note that when the pre-existing bets heavily favor outcome A or the track take is very large, the gross expected payoff of a bet on A is $W(A|A) < 1$ regardless of how many insiders bet on B . If so, no bets are then placed on outcome A in equilibrium. Condition (3.2) rules out such situations.

Equation (3.3) is derived from the indifference condition $\hat{p}_A W(A|A) = 1$ by using $m(A|A) = I(1 - G(\hat{p}_A|A))$ and $m(B|A) = IG(\hat{p}_B|A)$. As seen in Figure 3.1, this results in an inverse relationship between \hat{p}_A and \hat{p}_B . To see why this is the case, suppose by contradiction that instead \hat{p}_A and \hat{p}_B were to both rise, so that fewer insiders bet on A , and more insiders bet on B . Ceteris paribus, such a change makes it more attractive to bet on A , so that $W(A|A)$ rises. But \hat{p}_A is determined by the indifference among betting on A and not betting, so a rise in $W(A|A)$ implies a fall in \hat{p}_A , in contradiction with the

initial supposition. Similarly, the indifference condition $(1 - \hat{p}_B) W(B|B) = 1$ results in the downward sloping relationship (3.4). The proof of the proposition establishes that these curves cross precisely once, as illustrated in Figure 3.1.

3.2. Favorite-Longshot Bias

The market odds ratio for horse x is defined as the net return to a bet on x if x wins, i.e. $W(x|x) - 1$. The implied market probability for horse x is $(1 - \tau) / W(x|x)$, equal to the fraction of bets placed on it. According to a simple formulation of the market efficiency hypothesis, this market probability should aggregate market beliefs into an unbiased estimator of the horse's true probability of winning.

In order to test this proposition, empirical investigations of horse-race betting have typically proceeded by pooling data from many races. The outcomes of the races to which horses participate are used to estimate their empirical probability of winning depending on their market odds. The oft-observed favorite-longshot bias reveals that the greater the implied market probability of horse x , the greater the empirical average return to a dollar bet on x . Market probabilities thus understate the winning chances of favorites, and overstate the winning chances of longshots.

In accordance with this empirical approach, we compute the market implied probability for horse x in our equilibrium and relate it to its true chance of winning, conditional on the information contained in the resulting distribution of bets. Since higher private beliefs are more frequent when the true outcome is A (i.e., $G(p|B) > G(p|A)$ for all $0 < p < 1$), in equilibrium each individual bets more frequently on outcome x in state x . Therefore, in aggregate we have $m(A|A) > m(A|B)$ and $m(B|B) > m(B|A)$. This implies that horse x has a higher market probability (i.e., is more favored) when state x is true, and that the amounts $(m(A|x), m(B|x))$ bet by the insiders fully reveal x . Upon observation of $(m(A|x), m(B|x))$, outcome x is revealed to be true with probability one. The implied market probabilities, on the other hand, are never so extreme, since we always have $1 - \tau < W(x|x) < \infty$ and therefore $(1 - \tau) / W(x|x) \in (0, 1)$. We conclude that the equilibrium outcome of our model exhibits the favorite-longshot bias.

Proposition 2 *In the unique Bayes-Nash equilibrium of the last-period game, there is an extreme favorite-longshot bias. The insiders' bets $(m(A|x), m(B|x))$ reveal the true*

winner, and although horse x is more of a favorite ($m(x|x) > m(x|\neg x)$) when it wins, the market implied probability for the winner is less than one.

In this model, the favorite-longshot bias takes an extreme form due to the fact that there is a continuum of bettors, so that the law of large numbers applies and the final distribution of bets is deterministic. But the logic of the result is much more general, as shown in our companion paper Ottaviani and Sørensen (2004). In that paper, we investigate more generally the conditions for the occurrence of the favorite-longshot bias in the last round of betting in a model with a finite number of players. We show that the sign and extent of the favorite-longshot bias depends on the interaction of noise and information.¹⁸ As the number of bettors increases, the realized market odds contain more information and less noise. For any fixed market odds, the posterior odds are then more extreme and so the favorite-longshot bias is more pronounced. In the model with a continuum of insiders presented in this paper instead, noise is absent so that the favorite-longshot bias always arises. This happens more generally with a large enough number of bettors.

The result hinges on the fact that in a Bayes-Nash equilibrium, each insider does not know the total amount wagered by the other insiders. In a rational expectations equilibrium (REE), it is assumed instead that bettors can adjust their actions until they are satisfied with their bet, given their knowledge of the aggregate distribution of bets.¹⁹ Notice that an REE must be perfectly revealing in this setting,²⁰ so that in the REE all insiders bet on the winner and we have $m(A|A) = m(B|B) = N$.²¹ In our Bayes-Nash equilibrium, some insiders bet instead on the longshot, and not enough insiders bet on the favorite. As a result, the market implied probabilities are driven less towards the truth than in the REE. We conclude that our Bayes-Nash equilibrium results in a stronger

¹⁸In the presence of a few bettors with little private information, posterior odds are close to prior odds, even with extreme market odds, so that deviations of market odds from prior odds are mostly due to the noise contained in the signal. In that case, the market odds tend to be more extreme than the posterior odds, resulting in a reverse favorite-longshot bias. This explains why the reverse favorite-longshot bias is observed in lotto games, in which there is no private information.

¹⁹But in reality the aggregate amounts bet are observable only after all bets have been placed. REE models are not well suited to studying the performance of the trading structures of markets, because they assume that traders have more information than is actually available to them.

²⁰To see this, note that in an REE bettors also employ threshold strategies. If the state were not revealed, a positive fraction of them would bet on either horse (as above), and the amounts would then reveal the true outcome.

²¹Note that the insiders cannot completely remove the favorite-longshot bias even in the REE, due to their limited wealth. See Manski (2004) for an REE model of the Iowa Electronic Market, in which a deviation from the efficient market hypothesis arises due to limited wealth of traders.

favorite-longshot bias than the corresponding REE.

Potters and Wit (1996) have formulated the closest antecedent to our informational explanation for the favorite-longshot bias in parimutuel markets. In their as well as in our model, the favorite-longshot bias arises as a deviation from the rational expectations equilibrium. In Potters and Wit's model, the privately informed bettors are given the chance to adjust their bets at the final market odds, but they ignore the information contained in the bets.²² In our model instead, bettors fully understand the informational link, but are not allowed to adjust their bets after observing the final market odds. If the market closes immediately after the informed bets are placed, the market's tâtonnement process cannot incorporate this private information and reach a rational expectations equilibrium.

The extensive literature on the favorite-longshot bias contains a number of alternative explanations.²³ These explanations can be broadly classified into two groups, depending on whether they are based on the preferences of bettors or on the market microstructure. In the first group, Griffith (1949) suggested that individuals subjectively ascribe too large probabilities to rare events, while Weitzman (1965) and Ali (1977) hypothesized that individual bettors are risk loving, and so are willing to give up a larger expected payoff when assuming a greater risk (longer odds).

The explanations based on the market rules apply to either parimutuel or fixed odds betting. For the specific case of parimutuel markets, Isaacs (1953) noted that an informed monopolist bettor would not bet until the marginal bet has zero value, while Hurley and McDonough (1995) argued that a sizeable track take and the inability to place negative bets limits the amount of arbitrage by bettors with superior information, and so tends to result in relatively too few bets placed on the favorites. For fixed odds betting markets, Shin (1991 and 1992) explained the favorite-longshot bias as the response of an uninformed bookmaker to private information possessed by insider bettors. While Shin argues that a monopolistic bookmaker sets odds with a favorite-longshot bias in order to limit the subsequent losses to the better informed insiders, we derive the similar bias in a parimutuel market as the result of bets placed simultaneously.²⁴

²²Ali's (1977) Theorem 2 also features bettors with diverse beliefs who ignore others' information.

²³For a more extensive review of these explanations, see the surveys by Hausch and Ziemba (1995), Sauer (1998), and Jullien and Salanié (2002).

²⁴See Ottaviani and Sørensen (2004) for a comparison of the favorite longshot-bias arising in parimutuel and fixed-odds markets.

3.3. Comparative Statics

An increase in the track take directly makes rational betting less attractive, thereby causing the equilibrium thresholds to become more extreme.

Proposition 3 *Assuming that (3.2) holds, a marginal increase τ implies that \hat{p}_A strictly increases and \hat{p}_B strictly decreases, and that $W(A|A)$ and $W(B|B)$ both strictly decrease.*

Proof. See the Appendix. □

In a symmetric setting, we can derive a further comparative statics result. If the number of insiders increases (holding the outsider amounts $n(A), n(B)$ fixed), their bets have a greater impact on the market odds, tending in itself to make informed betting less attractive for individuals with a given signal. The equilibrium must have more extreme thresholds. More extreme indifference thresholds imply that the winner's odds ratio $W(x|x) - 1$ is lower, and thus the favorite-longshot bias is reduced. However, even as I grows arbitrarily large, the unique equilibrium remains interior, implying that $W(x|x) > 1$. The market implied probability $(1 - \tau) / W(x|x)$ can thus never exceed $1 - \tau$, since the track take prevents the informed population from fully correcting the odds resulting from the outsiders.

Proposition 4 *Assume that the distribution of private posterior beliefs is symmetric, that the initial bets are symmetric $n(A) = n(B) \equiv n > 0$, and that $0 < \tau < 1/2$. The unique equilibrium of Proposition 1 satisfies $\hat{p}_A = 1 - \hat{p}_B \in (1/2, 1)$. The threshold \hat{p}_A is increasing in τ and decreasing in n/I . The market odds become more extreme and the favorite-longshot bias is reduced when either n/I or τ is decreased.*

Proof. See the Appendix. □

The symmetric setting has the appealing property that the initial market belief in outcome A , $n(A) / (n(B) + n(A))$, is equal to the prior belief $q = 1/2$. A priori, then, the market odds are correct, and there is no scope for betting on the basis of public information alone. Nevertheless, privately informed individuals can profit from betting. In the symmetric model we have $m(A|B) = m(B|A) < m(B|B) = m(A|A)$, so the final implied market probabilities satisfy $(n + m(A|A)) / (n + n + m(B|A) + m(A|A)) > 1/2 > (n + m(A|B)) / (n + n + m(B|B) + m(A|B))$. When the market's implied probability of

an outcome exceeds $1/2$, but remains well below 1, the true (and empirical) probability of the outcome is 1. The favorite-longshot bias is evident.

Example. In the linear signal example with fair prior ($q = 1/2$), balanced pre-existing bets ($n(A) = n(B) \equiv n$), and track take $\tau \leq 1/2$, the unique symmetric-policy Nash equilibrium has an explicit expression, with cutoff belief

$$\hat{p}_A = \frac{(1 - \tau)(1 + n/I) - \sqrt{(1 + n/I)(\tau^2 + (1 - \tau)^2 n/I)}}{(1 - 2\tau)} \in [1/2, 1).$$

4. Late Betting with Private Information

Having characterized the equilibrium in the last period, we now turn to the analysis of the full dynamic game, in which insiders can bet in any period between $t = 1$ and $t = T$. A behavior strategy for a bettor with a given privately observed signal specifies how much remaining wealth to bet on either horse if any, after each publicly observed history. A perfect Bayesian equilibrium specifies a behavior strategy for each bettor, such that every bettor's strategy is optimal given the other bettors' strategies. Perfection requires that the continuation strategies should again constitute a perfect Bayesian equilibrium of the remaining game after any publicly observed tote board history, given rationally updated beliefs.

Proposition 5 *Assume that (3.2) holds. Then:*

- (i) *In all perfect Bayesian equilibria, the total amounts bet by the insiders are equal to those implied by Proposition 1.*
- (ii) *There exists a perfect Bayesian equilibrium in which all betting is postponed to the last period.*
- (iii) *If the thresholds of Proposition 1 satisfy $\hat{p}_B \leq q \leq \hat{p}_A$, this equilibrium is unique.*

Proof. See the Appendix. □

Proposition 5 concords with Asch, Malkiel and Quandt's (1982) empirical finding that late changes in odds predict the finishing order very well. As they argued informally, "bettors who have inside information would prefer to bet late in the period so as to minimize the time that the signal is available to the general public." Our theory suggests

that both the favorite-longshot bias and late informed betting can be ascribed to the presence of private information.²⁵

The proof of (i) also establishes that no information can be revealed by early bets on the equilibrium path. Notice that condition $\hat{p}_B \leq q \leq \hat{p}_A$ required for result (iii) is always satisfied when the model is symmetric, by Proposition 4. If this condition fails, say $\hat{p}_A < q$, then early betting could take place by insiders with private beliefs $p \in [p_1, p_2]$ where $\hat{p}_A < p_1 < q < p_2$ and $G(p_2|A) - G(p_1|A) = G(p_2|B) - G(p_1|B)$. Such bets do not reveal any information, and are of no consequence for the final distribution of bets, according to (i).

Our analysis of the timing game employs some simplifying assumptions. First, the proof of the proposition is somewhat facilitated by the continuum population assumption, in that a single player's deviation cannot be observed at all. However, the logic of this late timing result is much stronger. If an early bettor on horse A signals favorable information for this horse, then later bettors will find horse A more attractive and horse B less attractive. But from (3.1), the early bettor's return is decreasing in $m(A|A)$ and increasing in $m(B|A)$, so he does not desire later bettors to follow his lead in this fashion.²⁶ The incentive of informed traders to postpone their bet to the last minute is thus driven by the fact that in parimutuel betting all trades are executed at the same final price.²⁷

Second, we have assumed that the tote board is updated after each period, so that at time t the total amounts wagered at times $1, \dots, t-1$ is publicly known. In reality, there might be a slight delay in the tote board. With a delay of $\tau > 1$ periods, our prediction is that informed betting takes place within the last τ betting rounds, when betting is essentially simultaneous.

Third, we have assumed that the post time T is certain and commonly known. In reality, there might be some uncertainty about T , and this may drive the insiders to start betting somewhat before the actual closing time. Other bettors would then be able

²⁵Alternatively, late informed betting could be due by the fact that more information becomes publicly available during the betting period for exogenous reasons. However, an explanation based on public information would not be able to account for the presence of the favorite longshot bias.

²⁶Moreover, postponing the bet gives the option of learning from the other insiders' bets (although this cannot happen in the equilibrium of our model). Generally speaking, the uncertainty about the final bet distribution decreases over time, so it is safer to bet later.

²⁷The National Thoroughbred Racing Association (2004) is concerned that last-minute betting is facilitated by new methods of off-track betting, and that last-minute odds changes drive away a significant number of bettors.

to react to this information, resulting in an outcome closer to the rational expectations equilibrium. Again, the tote board delay helps to hide the bets for τ periods, making it less imperative that bets are placed at the very last moment. Notice, also, that the informed bettors would not start betting until when they believe there is a positive chance of market closure. Indeed, a sizeable fraction of bets are placed slightly before betting is closed.

5. Early Betting with Market Power

As we have seen in the previous section, individual bettors have an incentive to bet late if they have private information. That incentive is similar to the one that operates in auctions with fixed deadlines (see e.g., Roth and Ockenfels 2002) and in pre-opening markets (Biais, Hillion and Spatt 1999 and Medrano and Vives 2001). In this section, we show that parimutuel payoffs introduce an additional incentive that operates in the opposite direction.

In order to isolate this new incentive, we turn to study a finite number of large bettors who are able to place bets of arbitrary non-negative size. A bettor who can make a sizable bet faces an adverse movement in the odds, and should consider this effect when deciding how much to bet. This market power channel introduces an incentive to bet early, before other bettors place their bet to one's detriment.

Following Hurley and McDonough (1995), assume that there is a finite number I of rational bettors who share the same information about the state. In our setup of Section 2, this is the degenerate case of no (or completely uninformative) private signals, so that $p = q = \Pr(x = A)$ always. The amounts bet by others cannot then reveal any information.

Assume that the common prior belief, the track take and pre-existing bets are such that $q(1 - \tau) > n(A) / (n(A) + n(B))$, i.e. the pre-existing market probability is so far below the prior that it is profitable to bet on A . If bettors $i = 1, \dots, I$ place the amounts m_1, \dots, m_I on outcome A , bettor i 's payoff is

$$U_i(m_i) = q(1 - \tau) \frac{n(A) + n(B) + \sum_{i=1}^I m_i}{n(A) + \sum_{i=1}^I m_i} m_i - m_i.$$

This game is a special version of Cournot's model of quantity competition. To see this, interpret the amount m_i as the quantity produced at constant marginal unit cost by firm

i. Let the market's inverse demand curve be equal to

$$p(m) = q(1 - \tau) \frac{n(A) + n(B) + m}{n(A) + m}, \quad (5.1)$$

where m is the aggregate quantity. Bettors suffer inframarginal losses from increasing their own bets and so cease betting before the price for the last unit bet is equal to the marginal cost.

With simultaneous competition with $I < \infty$, the market implied probability for outcome A , $(n(A) + m) / (n(A) + n(B) + m)$, is lower than $q(1 - \tau)$, which would result in zero profits. This is for the usual reason that demand is above marginal cost ($p(m) > 1$). In turn it is lower than q , so although we assume that the rational bettors are placing their money on this horse (in this sense a favorite), the true probability of its winning is greater than the market implied probability. This finding is again in line with the favorite-longshot bias (see also Isaacs 1953).

The I bettors play a dynamic Cournot game. As is well known, in the two-player Stackelberg game, the leader would bet a larger amount and earn greater profits than the follower. By increasing the bet size beyond the static Cournot outcome, the leader pushes the follower to reply with a smaller amount. When the timing is endogenous, one would intuitively expect the insiders to place their bets as soon as possible, in order to profit from the early mover's advantage.

Following Hamilton and Slutsky (1990), we consider an "extended game with action commitment", according to which each bettor first commits to the one period in which to bet, and in that period decides how much to bet as a function of the betting history until then. At period t , it can be observed how much was bet by whom in all previous periods $1, \dots, t-1$, excluding the current period t . In our setting, this timing assumption incorporates the fact that the tote board can be observed, in addition to three less appealing properties: (i) each bettor can bet only once, (ii) each bettor cannot change the bet timing in reaction to others' bets as the game proceeds, and (iii) each bettor can observe precisely the set of bettors who have already bet and are therefore no longer active. The following result is proved by applying Matsumura's (1999) Proposition 3:

Proposition 6 *With I informed bettors, there are two possible subgame perfect equilibrium outcomes. In the first equilibrium, all bettors bet at $t = 1$, when they play the static Cournot outcome. In the second kind, all but one bettor place bets at $t = 1$.*

Proof. See the Appendix. □

In every equilibrium, all (but at most one) bettors place simultaneous bets in the first period, immediately after receiving their public information.²⁸ Intuitively, market power gives an incentive to move early, in order to capture a good share of the money on the table. This can explain why a sizeable fraction of bets are placed well before betting is closed — our third puzzle.

6. Conclusion

In this paper, we have proposed explanations for both the timing of informative bets and the favorite-longshot bias in parimutuel markets. Our explanations are based on the equilibrium incentives of some rational bettors, who derive positive expected profits due to the presence of outsiders.

We have identified a scenario with many small bettors in which *all* informed bets are placed at the end of the betting period, but our insight applies provided that *some* informed bets are placed simultaneously at the end. We have shown that small bettors have an incentive to delay betting in order to hide their private information. As a result, the final market odds do not reflect the beliefs of the market but rather tend to be less extreme than the posterior belief based on the information revealed by the final distribution of bets. Essentially, it is unfair to use the final market odds to evaluate the rationality of the bettors, because these odds are not known when the bets are placed. We have also pointed out that large bettors have a tempering incentive to bet early on the basis of public information, in order to prevent other bettors from acting on the same information.²⁹

Our theoretical findings are compatible with experimental results recently obtained by Plott, Wit and Yang (2003) in laboratory parimutuel markets. Their experimental subjects were endowed with limited budgets and given private signals informative about the likelihood of the different outcomes. Subjects could place bets up to their budget before the random termination of the markets. Compared to our model, the presence of a random termination time gives bettors an additional incentive to move early in order to reduce the termination risk. Although Bayes' rule was explained to the experimental subjects, not

²⁸It can be shown that there is an equilibrium in which all bettors play the static Cournot outcome in the first period, even when the timing assumptions of Matsumura are relaxed.

²⁹The analysis of the interplay of these two opposing incentives is an interesting topic for future research.

all profitable bets were made and some favorite-longshot bias was observed. According to the logic of our theory, the market odds were not equalized to the posterior odds because some of the informed bettors possibly postponed placing their limited budget, gambling that the termination would happen later.

Our findings call for further empirical work in the area, in which controls are added for the market rules. The cross-country and cross-market variation in the extent of the favorite-longshot bias points to the relevance of the market rules in determining the behavior of participants on the supply and demand side. In this direction, our companion paper (Ottaviani and Sørensen 2004) compares the equilibria resulting in fixed odds and parimutuel markets and shows that the comparative statics of the resulting favorite-longshot bias depends crucially on the market structure. Persistent differences in the observed biases could be attributed to varying degrees of market participation, informational asymmetry, and degrees of randomness in the post time.³⁰

Our analysis has implications for the design of parimutuel hedging markets.³¹ As stressed by Economides and Lange (2001), the parimutuel system is particularly apt for trading contingent claims.³² A major advantage of these markets is that the intermediary managing the parimutuel market is not exposed to any risk. On the flip side, market participants are subject to risk on the terms of trade and might have incentives to delay their orders. If traders are small and have private information, as seems reasonable, they might trade late and place orders mostly on the basis of their limited information, without access to information revealed by other market participants.³³ A thorough investigation of the implications for the design of prediction markets is left to future research.³⁴

In conclusion, the market structure plays a key role for the aggregation of private

³⁰Interesting selection issues arise when different betting schemes (fixed odd and parimutuel) coexist and compete to attract bets, as in the UK. As also suggested by Gabriel and Marsden (1990) and Bruce and Johnson (2000), bettors might have different incentives to place their bets on the parimutuel system rather than with the bookmakers, depending on the quality of their information.

³¹Since October 2002, Deutsche Bank and Goldman Sachs have been hosting Parimutuel Derivative Call Auctions of options on economic statistics. See Baron and Lange (2003) for a report on the performance of these markets.

³²While in betting markets these outsiders often have recreational motives, in hedging markets their role could be played by market participants with private values from hedging certain risks.

³³Trade will instead be early if it is based on common information and individual traders have market power. But note that these conditions are rather undesirable for the other market participants.

³⁴The Iowa electronic market seems to offer a promising hybrid. There, the intermediary bears no risk, and yet the possibility of trading at continuously changing prices should give the right incentives to react early to private information.

information into prices.³⁵ Our analysis suggests that parimutuel markets are not conducive to strong market efficiency, due to the incentive of privately informed traders to postpone their trades. In regular financial markets (e.g., as modeled in Kyle's 1985 continuous auction model), every order to buy asset A tends to increase the price of A , thereby eroding the profitability of later buy orders for A . Competition among insiders would then drive them to trade as early as possible (Holden and Subrahmanyam 1992). The incentives to reveal information crucially depend on the market structure and might explain the long-term performance of different trading institutions.³⁶

³⁵See Wolfers and Zitzewitz (2004) for a broad introduction to the informational content of market-generated forecasts.

³⁶In a traditional price-taking market, a rational expectations equilibrium need not reflect all available information, if there is randomness in the noise demand (see e.g., Hellwig 1980).

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Appendix: Proofs

Proof of Proposition 1. Given a set of strategies of the continuum of opponents, every agent can calculate the deterministic amounts $m(y|x)$, and can thus calculate $W(x|x) > 0$. Now $U(A|p) = pW(A|A) - 1$ is strictly increasing in p , $U(0|p) = 0$ is constant, and $U(B|p) = (1-p)W(B|B)$ is strictly decreasing. It follows that every individual has the same optimal response, characterized by thresholds \hat{p}_A and \hat{p}_B . Since $U(A|0) = U(B|1) = -1 < 0 = U(0|p)$ we have immediately that $\hat{p}_A > 0$ and $\hat{p}_B < 1$. Due to the threshold rule, the amounts bet are $m(A|A) = I(1 - G(\hat{p}_A|A))$, $m(A|B) = I(1 - G(\hat{p}_A|B))$, $m(B|A) = IG(\hat{p}_B|A)$, and $m(B|B) = IG(\hat{p}_B|B)$. Now, if $\hat{p}_A = 1$, then $W(A|A) = (1 - \tau)(n(A) + n(B) + m(B|A))/n(A) \geq (1 - \tau)(n(A) + n(B))/n(A) > 1$ by assumption, so $U(A|1) > U(0|1)$, contradicting optimality of the threshold $\hat{p}_A = 1$. Thus $\hat{p}_A < 1$, and a similar argument establishes $\hat{p}_B > 0$.

We now show $\hat{p}_B < \hat{p}_A$. Suppose to the contrary that $\hat{p}_B = \hat{p}_A$. Indifference yields $\hat{p}_A W(A|A) = (1 - \hat{p}_A)W(B|B)$, solved by $\hat{p}_A = W(B|B)/(W(A|A) + W(B|B)) \in (0, 1)$. Since abstention is not preferred, $\hat{p}_A W(A|A) \geq 1$, or equivalently $1 \geq 1/W(A|A) + 1/W(B|B)$. This is $1 - \tau \geq (n(A) + m(A|A))/(n(A) + n(B) + m(A|A) + m(B|A)) + (n(B) + m(B|B))/(n(A) + n(B) + m(A|B) + m(B|B))$. We now show that this fails since the right-hand side strictly exceeds 1, which is equivalent to $m(A|A)m(B|B) - m(B|A)m(A|B) > n(A)(m(B|A) - m(B|B)) + n(B)(m(A|B) - m(A|A))$. This inequality follows since the left-hand side is positive and the right-hand side negative, by $m(A|A) > m(A|B) > 0$ and $m(B|B) > m(B|A) > 0$. These facts follow from the expressions for $m(y|x)$ and the property that $G(\hat{p}_A|A) < G(\hat{p}_A|B)$.

We now have $0 < \hat{p}_B < \hat{p}_A < 1$, and optimality of the threshold rule implies the indifference conditions $U(A|\hat{p}_A) = 0$ and $U(B|\hat{p}_B) = 0$. Simple algebra results in the conditions (3.3) and (3.4). Rewrite (3.3) as

$$G(\hat{p}_B|A) = \frac{n(A)}{I(1 - \tau)\hat{p}_A} - \frac{n(A) + n(B)}{I} + \left(\frac{1}{(1 - \tau)\hat{p}_A} - 1 \right) (1 - G(\hat{p}_A|A)). \quad (6.1)$$

The right-hand side is a continuous function of $\hat{p}_A \in (0, 1)$. It tends to infinity as \hat{p}_A tends to 0 and is equal to $n(A)/(I(1 - \tau)) - (n(A) + n(B))/I < 0$ at $\hat{p}_A = 1$ by assumption (3.2). Moreover, its derivative is $-(n(A) + I(1 - G(\hat{p}_A|A)))/(I(1 - \tau)\hat{p}_A^2) - (1/((1 - \tau)\hat{p}_A) - 1)g(\hat{p}_A|A) < 0$. It follows that for every $\hat{p}_B \in [0, 1]$ there is a unique

solution $\hat{p}_A \in (0, 1)$ to equation (3.3). Thus equation (3.3) defines an implicit function in the space $(\hat{p}_B, \hat{p}_A) \in [0, 1]^2$. By the implicit function theorem, this function is downward sloping with

$$\left. \frac{d\hat{p}_A}{d\hat{p}_B} \right|_{(3.3)} = - \frac{g(\hat{p}_B|A)}{\frac{n(A)+I(1-G(\hat{p}_A|A))}{I(1-\tau)\hat{p}_A^2} + \left(\frac{1}{(1-\tau)\hat{p}_A} - 1 \right) g(\hat{p}_A|A)} < 0.$$

Likewise, for every $\hat{p}_A \in [0, 1]$ there is a unique solution $\hat{p}_B \in (0, 1)$ to (3.4), and the set of solutions defines a downward sloping curve in the space of $(\hat{p}_B, \hat{p}_A) \in [0, 1]^2$ with

$$\left. \frac{d\hat{p}_B}{d\hat{p}_A} \right|_{(3.4)} = - \frac{g(\hat{p}_A|B)}{\frac{n(B)+IG(\hat{p}_B|B)}{I(1-\tau)(1-\hat{p}_B)^2} + \left(\frac{1}{(1-\tau)(1-\hat{p}_B)} - 1 \right) g(\hat{p}_B|B)} < 0.$$

Existence follows from the fact that the curve defined by (3.3) traverses continuously from the left side $\{0\} \times (0, 1) \subseteq [0, 1]^2$ to the right side $\{1\} \times (0, 1) \subseteq [0, 1]^2$ of the $[0, 1]^2$ square, while the curve defined by (3.4) traverses continuously from the bottom $(0, 1) \times \{0\} \subseteq [0, 1]^2$ to the top $(0, 1) \times \{1\} \subseteq [0, 1]^2$. See Figure 3.1.

Uniqueness follows from the fact that the (3.4)-curve is steeper than the (3.3)-curve wherever they cross. Namely,

$$\left. \frac{d\hat{p}_A}{d\hat{p}_B} \right|_{(3.3)} \left. \frac{d\hat{p}_B}{d\hat{p}_A} \right|_{(3.4)} < 1.$$

This inequality is equivalent to

$$1 < \left[\frac{n(A)+I(1-G(\hat{p}_A|A))}{I(1-\tau)\hat{p}_A^2 g(\hat{p}_A|B)} + \frac{1-(1-\tau)\hat{p}_A}{(1-\tau)\hat{p}_A} \frac{g(\hat{p}_A|A)}{g(\hat{p}_A|B)} \right] \left[\frac{n(B)+IG(\hat{p}_B|B)}{I(1-\tau)(1-\hat{p}_B)^2 g(\hat{p}_B|A)} + \frac{1-(1-\tau)(1-\hat{p}_B)}{(1-\tau)(1-\hat{p}_B)} \frac{g(\hat{p}_B|B)}{g(\hat{p}_B|A)} \right],$$

where all the terms are positive. The inequality therefore holds since

$$\begin{aligned} & \frac{1-(1-\tau)\hat{p}_A}{(1-\tau)\hat{p}_A} \frac{g(\hat{p}_A|A)}{g(\hat{p}_A|B)} \frac{1-(1-\tau)(1-\hat{p}_B)}{(1-\tau)(1-\hat{p}_B)} \frac{g(\hat{p}_B|B)}{g(\hat{p}_B|A)} \\ &= \frac{1-(1-\tau)\hat{p}_A}{(1-\tau)\hat{p}_A} \frac{\hat{p}_A}{1-\hat{p}_A} \frac{1-(1-\tau)(1-\hat{p}_B)}{(1-\tau)(1-\hat{p}_B)} \frac{1-\hat{p}_B}{1-(1-\hat{p}_B)} \geq 1 \end{aligned}$$

where we used $g(p|A)/g(p|B) = (p/(1-p))((1-q)/q)$, $\hat{p}_A \geq (1-\tau)\hat{p}_A$, and $1-\hat{p}_B \geq (1-\tau)(1-\hat{p}_B)$. \square

Proof of Proposition 3. As shown in the proof of Proposition 1 and illustrated in Figure 3.1, the equilibrium is determined by the thresholds $(\hat{p}_B, \hat{p}_A) \in (0, 1)^2$ at which the

two downward sloping curves defined by (3.3) and (3.4) intersect. We have also noticed that the (3.4)-curve is steeper than the (3.3)-curve at the intersection.

Consider the effect of an increase in τ on the equilibrium. For any $\hat{p}_A \in (0, 1)$, the right-hand side of (6.1) is strictly increased. Thus, the left-hand side must strictly increase, in order to equilibrate. Thus, the (3.3)-curve shifts outwards: for any \hat{p}_A , the corresponding \hat{p}_B is strictly greater. Similarly, the (3.4)-curve shifts inwards: for any \hat{p}_B , the corresponding \hat{p}_A is strictly smaller. Since the steeper curve shifts inwards, and the flatter curve shifts outwards, we can conclude that the unique equilibrium must shift to the north-west in the (\hat{p}_B, \hat{p}_A) -space. Thus, \hat{p}_B strictly decreases, and \hat{p}_A strictly increases. Since the indifference conditions $\hat{p}_A W(A|A) = 1 = (1 - \hat{p}_B) W(B|B)$ continue to hold after the change, we can also conclude that $W(A|A)$ and $W(B|B)$ both strictly decrease. \square

Proof of Proposition 4. Using the assumptions, it is easy to verify that if the pair (\hat{p}_B, \hat{p}_A) solves (3.3) and (3.4), then $(1 - \hat{p}_A, \hat{p}_B)$ also solves. Since the solution is unique by Proposition 1, the equilibrium must satisfy $\hat{p}_A = 1 - \hat{p}_B$. Given this, condition (3.4) reduces to condition (3.3), and either condition can be rewritten as

$$(1 - \tau) \hat{p}_A = \frac{n/I + 1 - G(\hat{p}_A|A)}{2B/I + 1 - G(\hat{p}_A|A) + 1 - G(\hat{p}_A|B)}. \quad (6.2)$$

The right-hand side of (6.2) is continuous in \hat{p}_A . At $1/2$ it strictly exceeds the left-hand side, while the opposite is true at 1 . Thus the unique solution belongs to $(1/2, 1)$.

The left-hand side of (6.2) is strictly increasing in \hat{p}_A , while, at any solution, the right-hand side is a weakly decreasing function of \hat{p}_A . To see the latter claim, take the logarithm of the right-hand side, differentiate and use symmetry of G to arrive at the desired inequality

$$\frac{n/I + 1 - G(\hat{p}_A|A)}{n/I + 1 - G(\hat{p}_A|B)} \leq \frac{g(\hat{p}_A|A)}{g(\hat{p}_A|B)} = \frac{\hat{p}_A}{1 - \hat{p}_A},$$

i.e.,

$$\hat{p}_A \geq \frac{n/I + 1 - G(\hat{p}_A|A)}{2B/I + 1 - G(\hat{p}_A|A) + 1 - G(\hat{p}_A|B)},$$

which is implied by (6.2).

An increase in τ has a negative direct effect on the left-hand side of (6.2), so it results in an increase in \hat{p}_A . In turn, the market odds on the right-hand side is decreased. An increase in n/I reduces the right-hand side, so \hat{p}_A falls. Since the left-hand side falls, the market odds ratio on the right-hand side also falls. \square

Proof of Proposition 5. (i) First, we show that in equilibrium no information about x can be revealed by the observable history of past bets before T . Namely, the insiders' bets placed in any period $t < T$ must satisfy $m(A|A) = m(A|B)$ and $m(B|A) = m(B|B)$. Otherwise, the true state x would be fully revealed in the following period $t + 1$ and all the remaining bettors would bet on x , either until there are no more insiders or until $W(x|x) = 1$. Every bettor at t with an interior belief $p \in (0, 1)$ is therefore strictly better off postponing the bet — for the bet is on the wrong outcome giving a loss of -1 , or the bet returns the same as when postponing. But such a profitable deviation contradicts equilibrium.

We have therefore established that the game must proceed deterministically until period T , since the amounts bet by the insiders are independent of the true state. When the game reaches period T , the prior belief is still q . Now, the total amounts placed by insiders before and at time T must satisfy the characterization of the simultaneous equilibrium given in Proposition 1. For individual rationality in the last period again implies that there are thresholds \check{p}_B, \check{p}_A , such that individuals who did not bet before the final period will bet on B if $p < \check{p}_B$, bet on A if $p > \check{p}_A$, and otherwise abstain. Anyone who bet on horse B before time T must also have $p \leq \check{p}_B$, for the game proceeded deterministically, and at belief \check{p}_B there is zero expected return to the bet on B . Likewise, early bettors on A must have $p \geq \check{p}_A$. But then we have established that the equilibrium must be in common threshold strategies, and Proposition 1 characterized the only such equilibrium.

(ii) The following strategy profile constitutes such a perfect Bayesian equilibrium. After any history, all remaining bettors postpone their betting to the last period and play then the simultaneous Bayes-Nash equilibrium. The first observation of early insider bets makes beliefs shift to certainty that the most-bet horse is the winner; as long as both horses have received equal amount of insider bets, beliefs are unchanged.

To show that this is an equilibrium, consider any public history at time $t < T$. No individual player ever loses from postponing to the last period, for an earlier bet from the marginal player is too small to influence the beliefs of others through the tote board. Thus, the behavior of all other players is unaffected.

(iii) Suppose that bettors with private beliefs in the positive-measure set $K \subseteq [0, 1]$ bet on horse A at some period $t < T$. Since these bettors should eventually bet as proved in (i), then $K \subseteq [\hat{p}_A, 1]$. In addition, these bets cannot reveal any information, so that $m(A|B) =$

$I \Pr(K|B) = I \Pr(K|A) = m(A|A)$. But this equality is impossible, unless q belongs to the convex hull of K . To see this note that if $\hat{p}_A \geq q$, we have $\Pr(K|B) = \int_K g(p|B) dp = \int_K ((1-p)/p)(q/(1-q)) g(p|A) dp < ((1-\hat{p}_A)/\hat{p}_A)(q/(1-q)) \int_K g(p|A) dp \leq \Pr(K|A)$, leading to a contradiction. Likewise, there cannot be a positive measure of betting on horse B . \square

Proof of Proposition 6. This follows directly from Proposition 3 of Matsumura (1999), once we verify the three conditions about two-stage betting games with exogenous timing and arbitrary pre-existing bets. To analyze this two-stage game, suppose that I_1 bettors are leaders while I_2 are followers. Let $m^1(x)$ and $m^2(x)$ denote the total amounts bet by the leaders and followers, respectively. The three conditions are:

1. There exists a pure strategy equilibrium and the equilibrium is unique.
2. If the number of followers is one, this follower strictly prefers the Cournot outcome to the follower's outcome.
3. If the number of leaders is one, this leader strictly prefers the leader's outcome to the Cournot outcome.

The verification of these conditions involves straightforward computations available on request from the authors.

Omitted Details for Proof of Proposition 6

Not Intended for Publication

Here we verify Matsumura's three conditions.

(1) Consider the simultaneous game of the followers. We concentrate on the case where $q(1 - \tau) > (n(A) + m^1) / (n(A) + n(B) + m^1)$, for otherwise the leaders have bet irrationally too much. It is standard to show that there is a unique equilibrium of this simultaneous Cournot game. All followers bet the same amount m^2/I_2 , determined by the first order condition

$$q(1 - \tau) \left[\frac{n(A) + n(B) + m^1 + m^2}{n(A) + m^1 + m^2} - \frac{n(B)m^2/I_2}{(n(A) + m^1 + m^2)^2} \right] = 1. \quad (6.3)$$

In equilibrium, $q(1 - \tau) > (n(A) + m^1 + m^2) / (n(A) + n(B) + m^1 + m^2)$ so that all bettors earn a positive profit.

The leaders play a simultaneous betting game, taking into account how m^2 will respond to m^1 . In equilibrium, all bets belong to the open range $(0, M)$, where M is the competitive amount defined by $q(1 - \tau) = (n(A) + M) / (n(A) + n(B) + M)$. The necessary first order condition for an arbitrary leader's amount m_i^1 is then

$$q(1 - \tau) \left[\frac{n(A) + n(B) + m^1 + m^2}{n(A) + m^1 + m^2} - \frac{n(B)m_i^1}{(n(A) + m^1 + m^2)^2} \left(1 + \frac{dm^2}{dm^1} \Big|_{(6.3)} \right) \right] = 1.$$

Notice the implication that $1 + dm^2/dm^1|_{(6.3)} > 0$, since otherwise the left-hand side would exceed $q(1 - \tau)(n(A) + n(B) + m^1 + m^2) / (n(A) + m^1 + m^2)$ which strictly exceeds 1 when betting is profitable. Thus, holding m^1 fixed, the left-hand side is a strictly decreasing function of m_i^1 . It follows that all leaders optimally respond with the same quantity $m_i^1 = m^1/I_1$, so that the first order condition reduces to

$$q(1 - \tau) \left[\frac{n(A) + n(B) + m^1 + m^2}{n(A) + m^1 + m^2} - \frac{n(B)m^1/I_1}{(n(A) + m^1 + m^2)^2} \left(1 + \frac{dm^2}{dm^1} \Big|_{(6.3)} \right) \right] = 1. \quad (6.4)$$

It is now straightforward, if tedious, to verify that there is a unique solution (m^1, m^2) to equations (6.3) and (6.4). Matsumura's first condition is satisfied.

(2) The equilibrium amount m^C of the simultaneous Cournot game with I players is the solution in m^2 of equation (6.3) with $m^1 = 0$, i.e.

$$q(1 - \tau) \left[1 + \frac{n(B)(n(A) + \frac{I-1}{I}m^C)}{(n(A) + m^C)^2} \right] = 1. \quad (6.5)$$

When there is one follower, denote by m^L the total amount bet by the $n - 1$ leaders, and m^F the amount of the follower. From (6.3),

$$q(1 - \tau) \left[1 + \frac{n(B)(n(A) + m^L)}{(n(A) + m^L + m^F)^2} \right] = 1, \quad (6.6)$$

and from (6.4),

$$q(1 - \tau) \left[1 + \frac{n(B) \left(n(A) + m^F + \frac{I-2}{I-1} m^L - \frac{1}{I-1} m^L \frac{dm^F}{dm^L} \Big|_{(6.6)} \right)}{(n(A) + m^L + m^F)^2} \right] = 1. \quad (6.7)$$

In this Cournot setting any bettor is better off in equilibrium if and only if the remaining players reduce their total amount. Thus, (2) follows once we verify that $m^L > \frac{I-1}{I} m^C$. If $m^F < m^L / (I - 1)$, then using (6.5) and (6.6),

$$\frac{n(A) + m^L}{(a_{-1} + \frac{I}{I-1} m^L)^2} < \frac{n(A) + m^L}{(n(A) + m^L + m^F)^2} = \frac{n(A) + \frac{I-1}{I} m^C}{(n(A) + m^C)^2}$$

and it follows that $m^L > \frac{I-1}{I} m^C$ as desired. We therefore prove $m^F < m^L / (I - 1)$. Equations (6.7) and (6.6) imply that $m^F = \left(1 + dm^F/dm^L \Big|_{(6.6)} \right) m^L / (I - 1)$ so we need only verify $dm^F/dm^L \Big|_{(6.6)} < 0$. But this follows directly from (6.6).

(3) Note that the leader is weakly better off than in the Cournot game, since the leader could choose to bet the same amount as in the Cournot game. To verify that the leader is strictly better off, it is enough to verify the leader changes decision. If not, the others would still produce a total of $\frac{I-1}{I} m^C$. The leader's first order condition, derived from (6.4), would then be

$$q(1 - \tau) \left[1 + \frac{n(B) \left(n(A) + \frac{I-1}{I} m^C - \frac{dm^2}{dm^1} \Big|_{(6.3)} \frac{m^C}{I} \right)}{(n(A) + m^C)^2} \right] = 1,$$

in contradiction with (6.5). □